

# A non-PSLQ route to BBP-type formulas\*

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## Abstract

BBP-type formulas are usually discovered experimentally, through computer searches. In this paper, however, starting with two simple generators, and hence without doing any computer searches, we derive a wide range of BBP-type formulas in general bases. Many previously discovered BBP-type formulas turn out to be particular cases of the formulas derived here.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Generators of BBP-type formulas</b>	<b>2</b>
<b>3</b>	<b>Arctangent formulas</b>	<b>3</b>
3.1	BBP-type formulas generated by $x = \pi/2$ in identity (2.4)	3
3.2	BBP-type formulas generated by $x = \pi/3$ in identity (2.4)	5
3.3	BBP-type formulas generated by $x = \pi/4$ in identity (2.4)	6
3.4	BBP-type formulas generated by $x = \pi/6$ in identity (2.4)	9

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<b>4</b>	<b>Logarithm formulas</b>	<b>10</b>
4.1	BBP-type formulas generated by $x = \pi/2$ in identity (2.5)	. . 11
4.2	BBP-type formulas generated by $x = \pi/3$ in identity (2.5)	. . 11
4.3	BBP-type formulas generated by $x = \pi/4$ in identity (2.5)	. . 12
4.4	BBP-type formulas generated by $x = \pi/6$ in identity (2.5)	. . 12
<b>5</b>	<b>Summary</b>	<b>13</b>

## 1 Introduction

The study of BBP-type (Bailey, Borwein and Plouffe, 1997) formulas has continued to attract attention, mainly because they facilitate digit extraction through a simple algorithm not requiring multiple-precision arithmetic (Bailey, 2013). Experimentally, these formulas are usually discovered by using Bailey and Ferguson’s PSLQ (Partial Sum of Squares – Lower Quadrature) algorithm (Ferguson *et. al.*, 1999) or its variations. A downside is that PSLQ and other integer relation finding schemes typically do not suggest proofs (Bailey, 2006). Formal proofs must be developed after the formulas have been discovered. There have been attempts in the past to give general formulas which include the proofs, as can be found, for example, in the following references: (Bellard, 1997), (Broadhurst, 1998) and (Adamchik and Wagon, 1996). In this paper we give two identities which generate a wide range of BBP-type formulas in arbitrary bases. Many BBP-type formulas that are known in the literature turn out to be mere particular instances of the more general formulas presented here.

## 2 Generators of BBP-type formulas

Consider the Taylor series expansion

$$-\ln(1 - z) = \sum_{k=1}^{\infty} \frac{z^k}{k}, \quad (2.1)$$

valid for  $|z| \leq 1$ ,  $z \neq 1$ . Choosing  $z = p \cos x + ip \sin x$  in (2.1), for real  $p$  and  $x$ , allows one to write

$$\begin{aligned} -\ln(1-z) &\equiv -\ln \left[ \sqrt{(1-2p \cos x + p^2)} \exp \left( i \arctan \left( \frac{-p \sin x}{1-p \cos x} \right) \right) \right] \\ &= -\frac{1}{2} \ln(1-2p \cos x + p^2) + i \arctan \left( \frac{p \sin x}{1-p \cos x} \right) \end{aligned} \quad (2.2)$$

and, using De Moivre theorem,

$$\sum_{k=1}^{\infty} \frac{z^k}{k} \equiv \sum_{k=1}^{\infty} \frac{(p \cos x + ip \sin x)^k}{k} = \sum_{k=1}^{\infty} \frac{p^k \cos kx}{k} + i \sum_{k=1}^{\infty} \frac{p^k \sin kx}{k}. \quad (2.3)$$

Equating real and imaginary parts of (2.2) and (2.3) leads to the following identities:

$$\arctan \left( \frac{p \sin x}{1-p \cos x} \right) = \sum_{k=1}^{\infty} \frac{p^k \sin kx}{k} \quad (2.4)$$

and

$$-\frac{1}{2} \ln(1-2p \cos x + p^2) = \sum_{k=1}^{\infty} \frac{p^k \cos kx}{k}. \quad (2.5)$$

In the rest of this paper we demonstrate that careful choices of  $p$  and  $x$  in (2.4) and (2.5) lead to interesting BBP-type series, for  $|p| < 1$ .

### 3 Arctangent formulas

#### 3.1 BBP-type formulas generated by $x = \pi/2$ in identity (2.4)

The choice  $x = \pi/2$  in (2.4) gives the identity

$$\arctan p = \sum_{k=1}^{\infty} \frac{p^k \sin(k\pi/2)}{k}. \quad (3.1)$$

Since

$$\sin\left(\frac{k\pi}{2}\right) = \begin{cases} 1 & \text{if } k = 1, 5, 9, 13, 17, \dots \\ 0 & \text{if } k = 2, 4, 6, 8, 10, \dots \\ -1 & \text{if } k = 3, 7, 11, 15, 19, \dots \end{cases},$$

the identity (3.1) can be written as

$$\arctan p = \sum_{k=0}^{\infty} p^{4k} \left[ \frac{p}{4k+1} - \frac{p^3}{4k+3} \right].$$

Setting  $p = 1/u$ , the above identity can be written as

$$u^3 \arctan \frac{1}{u} = \sum_{k=0}^{\infty} \frac{1}{u^{4k}} \left[ \frac{u^2}{4k+1} - \frac{1}{4k+3} \right],$$

which is a BBP-type formula if  $u^2$  is a positive integer. Thus, with  $u^2 = n$ , we have:

$$\sqrt{n} \arctan\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{n} \sum_{k=0}^{\infty} \frac{1}{(n^2)^k} \left[ \frac{n}{4k+1} - \frac{1}{4k+3} \right], \quad n \in \mathbb{Z}^+. \quad (3.2)$$

In the notation employed in the BBP Compendium (Bailey, 2013),

$$\sqrt{n} \arctan\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{n} P(1, n^2, 4, (n, 0, -1, 0)).$$

We note that the particular case  $n = 2$  is a base-4 version of formula (21) of the Compendium. To see this we write the base  $n^2$ , length 4 formula (3.2) as a base  $n^4$ , length 8 formula as follows:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(n^2)^k} \left[ \frac{n}{4k+1} - \frac{1}{4k+3} \right] &= \sum_{k \text{ even}} () + \sum_{k \text{ odd}} () \\ &= \sum_{k=0}^{\infty} \frac{1}{(n^2)^{2k}} \left[ \frac{n}{4(2k)+1} - \frac{1}{4(2k)+3} \right] + \sum_{k=0}^{\infty} \frac{1}{(n^2)^{2k+1}} \left[ \frac{n}{4(2k+1)+1} - \frac{1}{4(2k+1)+3} \right] \\ &= \frac{1}{n^2} \sum_{k=0}^{\infty} \frac{1}{(n^4)^k} \left[ \frac{n^3}{8k+1} - \frac{n^2}{8k+3} + \frac{n}{8k+5} - \frac{1}{8k+7} \right], \end{aligned}$$

so that in base  $n^4$ , length 8 we have

$$\begin{aligned}\sqrt{n} \arctan\left(\frac{1}{\sqrt{n}}\right) &= \frac{1}{n^3} \sum_{k=0}^{\infty} \frac{1}{(n^4)^k} \left[ \frac{n^3}{8k+1} - \frac{n^2}{8k+3} + \frac{n}{8k+5} - \frac{1}{8k+7} \right] \\ &= \frac{1}{n^3} P(1, n^4, 8, (n^3, 0, -n^2, 0, n, -1)).\end{aligned}\tag{3.3}$$

The particular case  $n = 2$  in (3.3) recovers formula (21) of the Compendium.

Similarly, the particular case  $n = 3$  in (3.2) is a base-9 length 4 version of formula (66) of the Compendium.

In general a formula with base  $b$  and length  $l$  can be rewritten as a formula with base  $b^r$  and length  $rl$  (Bailey, 2013).

Identity (3.1) can also be written as

$$\arctan p = p \sum_{k=0}^{\infty} \frac{(-p^2)^k}{2k+1},$$

which gives the alternating base  $n$  version of (3.2) as

$$\begin{aligned}\sqrt{n} \arctan\left(\frac{1}{\sqrt{n}}\right) &= \sum_{k=0}^{\infty} \frac{1}{(-n)^k} \left[ \frac{1}{2k+1} \right], \quad n \in \mathbb{Z}^+ \\ &= P(1, -n, 2, (1, 0)).\end{aligned}$$

### 3.2 BBP-type formulas generated by $x = \pi/3$ in identity (2.4)

Putting  $x = \pi/3$  in (2.4) gives

$$\arctan\left(\frac{p\sqrt{3}}{2-p}\right) = \sum_{k=1}^{\infty} \frac{p^k \sin(k\pi/3)}{k}.$$

Noting that

$$\sin\left(\frac{k\pi}{3}\right) = \frac{\sqrt{3}}{2} \begin{cases} 1 & k=1,2,7,8,13,14,\dots \\ 0 & k=0,3,6,9,12,15,\dots \\ -1 & k=4,5,10,11,16,17,\dots \end{cases},$$

we have the following identity:

$$\sqrt{3} \arctan \left( \frac{p\sqrt{3}}{2-p} \right) = \frac{3}{2} \sum_{k=0}^{\infty} (-1)^k p^{3k} \left[ \frac{p}{3k+1} + \frac{p^2}{3k+2} \right],$$

which is a BBP-formula if  $p = 1/\pm n$ ,  $n$  a positive integer. The choice  $p = 1/n$  leads to

$$n^2 \sqrt{3} \arctan \left( \frac{\sqrt{3}}{2n-1} \right) = \frac{3}{2} \sum_{k=0}^{\infty} \frac{1}{(-n^3)^k} \left[ \frac{n}{3k+1} + \frac{1}{3k+2} \right], \quad n \in \mathbb{Z}^+, \quad (3.4)$$

while the choice  $p = -1/n$  gives

$$n^2 \sqrt{3} \arctan \left( \frac{\sqrt{3}}{2n+1} \right) = \frac{3}{2} \sum_{k=0}^{\infty} \frac{1}{(n^3)^k} \left[ \frac{n}{3k+1} - \frac{1}{3k+2} \right], \quad n \in \mathbb{Z}^+. \quad (3.5)$$

That is

$$n^2 \sqrt{3} \arctan \left( \frac{\sqrt{3}}{2n-1} \right) = \frac{3}{2} P(1, -n^3, 3, (n, 1, 0))$$

and

$$n^2 \sqrt{3} \arctan \left( \frac{\sqrt{3}}{2n+1} \right) = \frac{3}{2} P(1, n^3, 3, (n, -1, 0)).$$

A particular case of (3.5) is formula (65) in the BBP Compendium, corresponding to  $n = 3$  here.  $n = 2$  in (3.4) also gives a formula that is equivalent to formula (18) in the Compendium.

### 3.3 BBP-type formulas generated by $x = \pi/4$ in identity (2.4)

$x = \pi/4$  in (2.4) gives

$$\arctan \left( \frac{p}{\sqrt{2}-p} \right) = \sum_{k=1}^{\infty} \frac{p^k \sin(k\pi/4)}{k}.$$

Observing that

$$\sin\left(\frac{k\pi}{4}\right) = \begin{cases} 1 & k=2,10,18,26,34,\dots \\ 1/\sqrt{2} & k=1,3,9,11,17,19,\dots \\ 0 & k=0,4,8,12,16,20,\dots \\ -1/\sqrt{2} & k=5,7,13,15,21,23,\dots \\ -1 & k=6,14,22,30,38,46,\dots \end{cases},$$

we obtain

$$\arctan\left(\frac{p}{\sqrt{2}-p}\right) = \sum_{k=0}^{\infty} p^{8k} \left[ \frac{p}{\sqrt{2}} \frac{1}{8k+1} + \frac{p^2}{8k+2} + \frac{p^3}{\sqrt{2}} \frac{1}{8k+3} - \frac{p^5}{\sqrt{2}} \frac{1}{8k+5} - \frac{p^6}{8k+6} - \frac{p^7}{\sqrt{2}} \frac{1}{8k+7} \right].$$

On setting  $p = \sqrt{2}/u$ , the above identity can be written as

$$u^7 \arctan\left(\frac{1}{u-1}\right) = \sum_{k=0}^{\infty} \frac{1}{(u/\sqrt{2})^{8k}} \left[ \frac{u^6}{8k+1} + \frac{2u^5}{8k+2} + \frac{2u^4}{8k+3} - \frac{4u^2}{8k+5} - \frac{8u}{8k+6} - \frac{8}{8k+7} \right], \quad (3.6)$$

while  $p = -\sqrt{2}/u$  gives

$$u^7 \arctan\left(\frac{1}{u+1}\right) = \sum_{k=0}^{\infty} \frac{1}{(u/\sqrt{2})^{8k}} \left[ \frac{u^6}{8k+1} - \frac{2u^5}{8k+2} + \frac{2u^4}{8k+3} - \frac{4u^2}{8k+5} + \frac{8u}{8k+6} - \frac{8}{8k+7} \right]. \quad (3.7)$$

Identities (3.6) and (3.7) are BBP-type series if  $u$  is an even integer. Thus, setting  $u = 2n$  in both identities, we obtain the following BBP-type formulas:

$$n^7 \arctan\left(\frac{1}{2n-1}\right) = \frac{1}{16} \sum_{k=0}^{\infty} \frac{1}{(16n^8)^k} \left[ \frac{8n^6}{8k+1} + \frac{8n^5}{8k+2} + \frac{4n^4}{8k+3} - \frac{2n^2}{8k+5} - \frac{2n}{8k+6} - \frac{1}{8k+7} \right], \quad n \in \mathbb{Z}^+ \quad (3.8)$$

and

$$n^7 \arctan\left(\frac{1}{2n+1}\right) = \frac{1}{16} \sum_{k=0}^{\infty} \frac{1}{(16n^8)^k} \left[ \frac{8n^6}{8k+1} - \frac{8n^5}{8k+2} + \frac{4n^4}{8k+3} - \frac{2n^2}{8k+5} + \frac{2n}{8k+6} - \frac{1}{8k+7} \right], \quad n \in \mathbb{Z}^+. \quad (3.9)$$

In the P-notation then,

$$n^7 \arctan\left(\frac{1}{2n-1}\right) = \frac{1}{16} P(1, 16n^8, 8, (8n^6, 8n^5, 4n^4, 0, -2n^2, -2n, -1, 0))$$

and

$$n^7 \arctan\left(\frac{1}{2n+1}\right) = \frac{1}{16} P(1, 16n^8, 8, (8n^6, -8n^5, 4n^4, 0, -2n^2, 2n, -1, 0)).$$

Formula (15) of the Compendium is a particular case of (3.8), with  $n = 1$ .

Adding (3.6) and (3.7), we obtain

$$u^7 \arctan\left(\frac{2u}{u^2-2}\right) = 2 \sum_{k=0}^{\infty} \frac{1}{(u/\sqrt{2})^{8k}} \left[ \frac{u^6}{8k+1} + \frac{2u^4}{8k+3} - \frac{4u^2}{8k+5} - \frac{8}{8k+7} \right],$$

which is a BBP-series only if  $u^2$  is an even integer. Thus, setting  $u^2 = 2n$  in the above identity, we obtain the BBP-type formula

$$n^3 \sqrt{2n} \arctan\left(\frac{\sqrt{2n}}{n-1}\right) = 2 \sum_{k=0}^{\infty} \frac{1}{n^{4k}} \left[ \frac{n^3}{8k+1} + \frac{n^2}{8k+3} - \frac{n}{8k+5} - \frac{1}{8k+7} \right],$$

that is

$$n^3 \sqrt{2n} \arctan\left(\frac{\sqrt{2n}}{n-1}\right) = 2P(1, n^4, 8, (n^3, 0, n^2, 0, -n, 0, -1, 0)).$$

The particular case  $n = 2$  corresponds to formula (8) in the Compendium.

Subtracting (3.7) from (3.6) gives a formula which is equivalent to (3.2) and therefore contains no new information.



### 3.4 BBP-type formulas generated by $x = \pi/6$ in identity (2.4)

With  $x = \pi/6$  in (2.4), we have

$$\arctan\left(\frac{2p + p^2\sqrt{3}}{4 - 3p^2}\right) = \sum_{k=1}^{\infty} \frac{p^k \sin(k\pi/6)}{k}.$$

Noting that

$$\sin\left(\frac{k\pi}{6}\right) = \begin{cases} 1 & k=3,15,27,39,51,\dots \\ \sqrt{3}/2 & k=2,4,14,16,26,28,\dots \\ 1/2 & k=1,5,13,17,25,29,37,41,\dots \\ 0 & k=6,12,18,24,30,36,\dots \\ -1/2 & k=7,11,19,23,31,35,\dots \\ -\sqrt{3}/2 & k=8,10,20,22,32,34,\dots \\ -1 & k=9,21,33,45,57,\dots \end{cases},$$

we obtain

$$\begin{aligned} & \arctan\left(\frac{2p + p^2\sqrt{3}}{4 - 3p^2}\right) \\ &= \sum_{k=0}^{\infty} (-p^6)^k \left[ \frac{1}{2} \frac{p}{6k+1} + \frac{\sqrt{3}}{2} \frac{p^2}{6k+2} + \frac{p^3}{6k+3} + \frac{\sqrt{3}}{2} \frac{p^4}{6k+4} + \frac{1}{2} \frac{p^5}{6k+5} \right]. \end{aligned}$$

$p = \sqrt{3}/u$  and  $p = -\sqrt{3}/u$  in the above identity yield the following series:

$$\begin{aligned} & u^5 \sqrt{3} \arctan\left(\frac{\sqrt{3}}{2u-3}\right) \\ &= \frac{3}{2} \sum_{k=0}^{\infty} \frac{1}{(-u^6/27)^k} \left[ \frac{u^4}{6k+1} + \frac{3u^3}{6k+2} + \frac{6u^2}{6k+3} + \frac{9u}{6k+4} + \frac{9}{6k+5} \right] \end{aligned}$$

and

$$\begin{aligned} & u^5 \sqrt{3} \arctan\left(\frac{\sqrt{3}}{2u+3}\right) \\ &= \frac{3}{2} \sum_{k=0}^{\infty} \frac{1}{(-u^6/27)^k} \left[ \frac{u^4}{6k+1} - \frac{3u^3}{6k+2} + \frac{6u^2}{6k+3} - \frac{9u}{6k+4} + \frac{9}{6k+5} \right], \end{aligned}$$

which are BBP-type series if  $u$  is a multiple of 3. Thus, with  $u = 3n$ , we obtain the following BBP-type series:

$$\begin{aligned}
& 27n^5\sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\frac{1}{2n-1}\right) \\
&= \frac{3}{2}\sum_{k=0}^{\infty}\frac{1}{(-27n^6)^k}\left[\frac{9n^4}{6k+1}+\frac{9n^3}{6k+2}+\frac{6n^2}{6k+3}+\frac{3n}{6k+4}+\frac{1}{6k+5}\right] \quad (3.10) \\
&= \frac{3}{2}P(1, -27n^6, 6, (9n^4, 9n^3, 6n^2, 3n, 1, 0))
\end{aligned}$$

and

$$\begin{aligned}
& 27n^5\sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\frac{1}{2n+1}\right) \\
&= \frac{3}{2}\sum_{k=0}^{\infty}\frac{1}{(-27n^6)^k}\left[\frac{9n^4}{6k+1}-\frac{9n^3}{6k+2}+\frac{6n^2}{6k+3}-\frac{3n}{6k+4}+\frac{1}{6k+5}\right] \quad (3.11) \\
&= \frac{3}{2}P(1, -27n^6, 6, (9n^4, -9n^3, 6n^2, -3n, 1, 0)).
\end{aligned}$$

Formula (66) of the Compendium is a particular case of formula (3.10), corresponding to setting  $n = 1$ .

Addition of (3.10) and (3.11) gives the following BBP-type series

$$\begin{aligned}
n^2\sqrt{n}\arctan\left(\frac{\sqrt{n}}{n-1}\right) &= \sum_{k=0}^{\infty}\frac{1}{(-n^3)^k}\left[\frac{n^2}{6k+1}+\frac{2n}{6k+3}+\frac{1}{6k+5}\right] \quad (3.12) \\
&= P(1, -n^3, 6, (n^2, 0, 2n, 0, 1, 0)).
\end{aligned}$$

Subtraction of (3.10) and (3.11) yields (3.4) and therefore does not give new information.

## 4 Logarithm formulas

Working in a similar fashion to that in the previous section, we present the following BBP-type formulas for logarithm.

#### 4.1 BBP-type formulas generated by $x = \pi/2$ in identity (2.5)

$$\begin{aligned}\ln\left(\frac{n+1}{n}\right) &= \frac{1}{n^2} \sum_{k=0}^{\infty} \frac{1}{(n^2)^k} \left[ \frac{n}{2k+1} - \frac{1}{2k+2} \right] \\ &= \frac{1}{n^2} P(1, n^2, 2, (n, -1)).\end{aligned}\tag{4.1}$$

$$\begin{aligned}\ln\left(\frac{n-1}{n}\right) &= -\frac{1}{n^2} \sum_{k=0}^{\infty} \frac{1}{(n^2)^k} \left[ \frac{n}{2k+1} + \frac{1}{2k+2} \right] \\ &= -\frac{1}{n^2} P(1, n^2, 2, (n, 1)).\end{aligned}\tag{4.2}$$

Addition of (4.1) and (4.2) gives

$$\begin{aligned}\ln\left(\frac{n-1}{n}\right) &= -\frac{1}{n} \sum_{k=0}^{\infty} \frac{1}{n^k} \left[ \frac{1}{k+1} \right] \\ &= -\frac{1}{n} P(1, n, 1, (1)).\end{aligned}\tag{4.3}$$

Formula (81) of the BBP Compendium is a particular case of (4.3), with  $n = 10$ .

Subtraction of (4.2) from (4.1) gives

$$\begin{aligned}\sqrt{n} \ln\left(\frac{\sqrt{n}+1}{\sqrt{n}-1}\right) &= 2 \sum_{k=0}^{\infty} \frac{1}{n^k} \left[ \frac{1}{2k+1} \right] \\ &= 2P(1, n, 2, (1, 0)).\end{aligned}\tag{4.4}$$

Formula (6) of the Compendium is a particular case of (4.4), with  $n = 4$ , while Formula (64) of the Compendium is another particular case, with  $n = 9$ .  $n = 2$  in (4.4) gives a length 2 version of formula (20) in the Compendium.

#### 4.2 BBP-type formulas generated by $x = \pi/3$ in identity (2.5)

$$\begin{aligned}\ln\left(\frac{n^2 - n + 1}{n^2}\right) &= -\frac{1}{n^3} \sum_{k=0}^{\infty} \frac{1}{(-n^3)^k} \left[ \frac{n^2}{3k+1} - \frac{n}{3k+2} - \frac{2}{3k+3} \right] \\ &= -\frac{1}{n^3} P(1, -n^3, 3, (n^2, -n, -2)).\end{aligned}$$

$$\begin{aligned}\ln\left(\frac{n^2+n+1}{n^2}\right) &= \frac{1}{n^3} \sum_{k=0}^{\infty} \frac{1}{(n^3)^k} \left[ \frac{n^2}{3k+1} + \frac{n}{3k+2} - \frac{2}{3k+3} \right] \\ &= \frac{1}{n^3} P(1, n^3, 3, (n^2, n, -2)).\end{aligned}$$

### 4.3 BBP-type formulas generated by $x = \pi/4$ in identity (2.5)

$$\begin{aligned}\ln\left(\frac{2n^2-2n+1}{2n^2}\right) &= -\frac{1}{2n^4} \sum_{k=0}^{\infty} \frac{1}{(-4n^4)^k} \left[ \frac{2n^3}{4k+1} - \frac{n}{4k+3} - \frac{1}{4k+4} \right] \\ &= -\frac{1}{2n^4} P(1, -4n^4, 4, (2n^3, 0, -n, -1)).\end{aligned}$$

$$\begin{aligned}\ln\left(\frac{2n^2+2n+1}{2n^2}\right) &= \frac{1}{2n^4} \sum_{k=0}^{\infty} \frac{1}{(-4n^4)^k} \left[ \frac{2n^3}{4k+1} - \frac{n}{4k+3} + \frac{1}{4k+4} \right] \\ &= \frac{1}{2n^4} P(1, -4n^4, 4, (2n^3, 0, -n, 1)).\end{aligned}$$

$$\begin{aligned}n \frac{\sqrt{n}}{\sqrt{2}} \ln\left(\frac{n+\sqrt{2}\sqrt{n}+1}{n-\sqrt{2}\sqrt{n}+1}\right) &= 2 \sum_{k=0}^{\infty} \frac{1}{(-n^2)^k} \left[ \frac{n}{4k+1} - \frac{1}{4k+3} \right] \\ &= 2P(1, -n^2, 4, (n, 0, -1, 0)).\end{aligned} \tag{4.5}$$

Note that  $n = 2$  in (4.5) gives a binary BBP-type formula for  $\log 5$ .

### 4.4 BBP-type formulas generated by $x = \pi/6$ in identity (2.5)

$$\begin{aligned}\ln\left(\frac{3n^2-3n+1}{3n^2}\right) &= -\frac{1}{27n^6} \sum_{k=0}^{\infty} \frac{1}{(-27n^6)^k} \left[ \frac{27n^5}{6k+1} + \frac{9n^4}{6k+2} - \frac{3n^2}{6k+4} - \frac{3n}{6k+5} - \frac{2}{6k+6} \right] \\ &= -\frac{1}{27n^6} P(1, -27n^6, 6, (27n^5, 9n^4, 0, -3n^2, -3n, -2)).\end{aligned}$$

$$\begin{aligned}\ln\left(\frac{3n^2+3n+1}{3n^2}\right) &= \frac{1}{27n^6} \sum_{k=0}^{\infty} \frac{1}{(-27n^6)^k} \left[ \frac{27n^5}{6k+1} - \frac{9n^4}{6k+2} + \frac{3n^2}{6k+4} - \frac{3n}{6k+5} + \frac{2}{6k+6} \right] \\ &= \frac{1}{27n^6} P(1, -27n^6, 6, (27n^5, -9n^4, 0, 3n^2, -3n, 2)).\end{aligned}$$

$$\begin{aligned}
n^2 \frac{\sqrt{n}}{\sqrt{3}} \ln \left( \frac{n + \sqrt{3}\sqrt{n} + 1}{n - \sqrt{3}\sqrt{n} + 1} \right) &= 2 \sum_{k=0}^{\infty} \frac{1}{(-n^3)^k} \left[ \frac{n^2}{6k+1} - \frac{1}{6k+5} \right] \\
&= 2P(1, -n^3, 6, (n^2, 0, 0, 0, -1, 0)).
\end{aligned} \tag{4.6}$$

The particular case  $n = 3$  of (4.6) is equivalent to the series obtained for  $\log 7$  in reference (Adamchik and Wagon, 1996).

## 5 Summary

Starting with two simple generators, we have derived a wide range of BBP-type formulas in general bases, namely,

$$\begin{aligned}
\sqrt{n} \arctan \left( \frac{1}{\sqrt{n}} \right) &= \frac{1}{n} P(1, n^2, 4, (n, 0, -1, 0)) \\
n^2 \sqrt{3} \arctan \left( \frac{\sqrt{3}}{2n-1} \right) &= \frac{3}{2} P(1, -n^3, 3, (n, 1, 0)) \\
n^2 \sqrt{3} \arctan \left( \frac{\sqrt{3}}{2n+1} \right) &= \frac{3}{2} P(1, n^3, 3, (n, -1, 0)) \\
n^7 \arctan \left( \frac{1}{2n-1} \right) &= \frac{1}{16} P(1, 16n^8, 8, (8n^6, 8n^5, 4n^4, 0, -2n^2, -2n, -1, 0)) \\
n^7 \arctan \left( \frac{1}{2n+1} \right) &= \frac{1}{16} P(1, 16n^8, 8, (8n^6, -8n^5, 4n^4, 0, -2n^2, 2n, -1, 0)) \\
n^3 \sqrt{2n} \arctan \left( \frac{\sqrt{2n}}{n-1} \right) &= 2P(1, n^4, 8, (n^3, 0, n^2, 0, -n, 0, -1, 0)) \\
9n^5 \sqrt{3} \arctan \left( \frac{1}{\sqrt{3}} \frac{1}{2n-1} \right) &= \frac{1}{2} P(1, -27n^6, 6, (9n^4, 9n^3, 6n^2, 3n, 1, 0)) \\
9n^5 \sqrt{3} \arctan \left( \frac{1}{\sqrt{3}} \frac{1}{2n+1} \right) &= \frac{1}{2} P(1, -27n^6, 6, (9n^4, -9n^3, 6n^2, -3n, 1, 0)) \\
n^2 \sqrt{n} \arctan \left( \frac{\sqrt{n}}{n-1} \right) &= P(1, -n^3, 6, (n^2, 0, 2n, 0, 1, 0))
\end{aligned}$$

$$\begin{aligned}
\ln\left(\frac{n+1}{n}\right) &= \frac{1}{n^2}P(1, n^2, 2, (n, -1)) \\
\ln\left(\frac{n-1}{n}\right) &= -\frac{1}{n^2}P(1, n^2, 2, (n, 1)) \\
\sqrt{n}\ln\left(\frac{\sqrt{n}+1}{\sqrt{n}-1}\right) &= 2P(1, n, 2, (1, 0)) \\
\ln\left(\frac{n^2+n+1}{n^2}\right) &= \frac{1}{n^3}P(1, n^3, 3, (n^2, n, -2)) \\
\ln\left(\frac{n^2-n+1}{n^2}\right) &= -\frac{1}{n^3}P(1, -n^3, 3, (n^2, -n, -2)) \\
\ln\left(\frac{2n^2-2n+1}{2n^2}\right) &= -\frac{1}{2n^4}P(1, -4n^4, 4, (2n^3, 0, -n, -1)) \\
\ln\left(\frac{2n^2+2n+1}{2n^2}\right) &= \frac{1}{2n^4}P(1, -4n^4, 4, (2n^3, 0, -n, 1)) \\
n\frac{\sqrt{n}}{\sqrt{2}}\ln\left(\frac{n+\sqrt{2}\sqrt{n}+1}{n-\sqrt{2}\sqrt{n}+1}\right) &= 2P(1, -n^2, 4, (n, 0, -1, 0)) \\
\ln\left(\frac{3n^2\pm 3n+1}{3n^2}\right) &= \pm\frac{1}{27n^6}P(1, -27n^6, 6, (27n^5, \mp 9n^4, 0, \pm 3n^2, -3n, \pm 2)) \\
n^2\frac{\sqrt{n}}{\sqrt{3}}\ln\left(\frac{n+\sqrt{3}\sqrt{n}+1}{n-\sqrt{3}\sqrt{n}+1}\right) &= 2P(1, -n^3, 6, (n^2, 0, 0, 0, -1, 0))
\end{aligned}$$

Many previously discovered BBP-type formulas turn out to be particular cases of the above formulas.

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